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Applications of Homotopy perturbation Method and Sumudu Transform for Solving Fractional Initial Boundary Value problem and Fractional KDV Equation

A. S. Abedl-Rady*, S. Z. Rida*, A. A. M. Arafa**, and H. R. Abedl-Rahim*
*Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt
**Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said, Egypt

***Abstract**: In this paper, we make use of the properties of the Sumudu transform to find the exact solution of fractional initial-boundary value problem (FIBVP) and fractional KDV equation. The method, namely, homotopy perturbation Sumudu transform method, is the combination of the Sumudu transform and the HPM using He's polynomials. This method is very powerful and professional techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. We also present two different examples to illustrate the preciseness and effectiveness of this method.*

***Keywords**: Sumudu transform, Homotopy perturbation Sumudu transform method, He's polynomials, Fractional linear inhomogeneous KDV equation, Fractional initial boundary value problem.*

I. Introduction

Fractional differential equations have attracted much attention, recently, see, for instance [1–5]. This is mostly due to the fact that fractional calculus provides an efficient and excellent instrument for the description of many practical dynamical phenomena arising in engineering and scientific disciplines such as, physics, chemistry, biology, economy, viscoelasticity, electrochemistry, electromagnetic, control, porous media, and many more, see, for example, [6–9]. In the present paper, we propose a new method called Homotopy perturbation Sumudu transform method (HPSTM) for solving the linear and initial value problems. It is worth mentioning that the proposed method is an elegant combination of the Sumudu transformation, the Homotopy perturbation method and He's Polynomials and is mainly due to Ghorbani [10,11].The use of He's polynomials in the nonlinear term was first introduced by Ghorbani [10,11]. The Proposed algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of this method is its capability of combining two powerful methods for obtaining exact and numerical solutions for nonlinear dispersive equations. This paper considers the effectiveness of the Homotopy Perturbation Sumudu transform method (HPSTM) in solving fractional initial-boundary value (FIBVP) and fractional KDV *equation*.

Basic Definitions

Definition 1:

The Riemann-Lowville fractional integral operator of order $\alpha > 0$, of a function $f(t) \in c_p$, and $p \geq -1$, is defined as [12]:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, (\alpha > 0),$$

$$I^0 f(t) = f(t).$$

For the Riemann- Liouville fractional integral we have

$$I^\alpha . t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$$

Definition 2:

In early 90's [13] Watauga introduced a new integral transform, named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform is defined over the set of functions:

$$A = \left\{ f(t) : \exists M \tau_1, \tau_2 > 0, | \; f(t) \; | < M e^{|t|/\tau_j}, if \; t \in (-1)^j \times [0, \infty) \right\}$$

by the following formula:

$$G(u)=S[f(t);u]=\int_0^\infty f(ut)e^{-t}dt,u\in(-\tau_{1,},\tau_2).$$

2.1 Some special properties of the Sumudu transform are as follows:

2.1.1 $S[1]=1$;

2.1.2 $S[\frac{t^n}{\Gamma(n+1)}]=u^n;n>0$

2.1.3 $S[e^{at}]=\frac{1}{1-au}$;

2.1.4 $S[af(t)+bg(t)]=aS[f(t)]+bS[g(t)]$

Other properties of the Sumudu transform can be found in [14].

Definition 3:

The Sumudu transform of the Caputo fractional derivative is defined as follows [15]:

$S[D_t^\alpha f(t)]=u^{-\alpha}S[f(t)]-\sum_{k=0}^{n-1}u^{-\alpha+k}f^{(k)}(0+),(n-1<\alpha\le n)$ where $G(u)=S[f(t)]$.

Definition 4:

The Mittag-Leffler function $E_\alpha(z)$ with $\alpha>0$ is defined by the following series representation, valid in the whole complex plane [16];

$$E_\alpha(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(\alpha k+1)}$$

II. Homotopy Perturbation Sumudu Transform Method (HPSTM).

To illustrate the basic idea of this method, we consider the following nonlinear fractional differential equation:

$$D_t^\alpha U(x,t)+L(U(x,t))+N(U(x,t))=q(x,t),t>0,0<\alpha<1 \quad (1)$$

Subject to initial condition:

$U(x,0)=f(x)$

where $D_t^\alpha=\partial^\alpha/\partial^\alpha$ is the fractional Caputo derivative of the function $U(x,t)$,

L is the linear differential operator, *N* is the nonlinear differential operator, and $q(x,t)$ is the source term.

Now, applying the Sumudu transform on both sides of (1) we have:

$$S[D_t^\alpha U(x,t)]+S[L(U(x,t))]+S[N(U(x,t))]=S[q(x,t)]. \quad (2)$$

Using the differential property of Sumudu transform, we have

$$S[U(x,t)]=f(x)-u^\alpha S[L(U(x,t))+N(U(x,t))]+u^\alpha S[q(x,t)]. \quad (3)$$

Operating with Sumudu inverse on both sides of (3)

$$U(x,t)=Q(x,t)-S^{-1}[u^\alpha S[L(U(x,t))+N(U(x,t))]]. \quad (4)$$

where $Q(x,t)$ represents the term arising from the source term and the prescribed initial conditions.

Now, applying the classical homotopy perturbation technique, the solution can be expressed as a power series in p as given below:

$$U(x,t)=\sum_{n=0}^\infty p^nU_n(x,t), \quad (5)$$

where the homotopy parameter p is considered as a small parameter $p\in[0,1]$.

We can decompose the nonlinear term as:

$$NU(x,t)=\sum_{n=0}^\infty p^nH_n(U), \quad (6)$$

where H_n are He's polynomials of $U_0(x,t),U_1(x,t),U_2(x,t),...., U_n(x,t)$ [15-17]

and it can be calculated by the following formula:

$$H_n(U_0(x,t),U_1(x,t),U_2(x,t),....,U_n(x,t)) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}[N(\sum_{i=0}^{\infty} p^i U_i)]_{p=0}, n = 0,1,2,... \tag{7}$$

By substituting (5) and (6) and using HPM [15] we get:

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = Q(x,t) - p(S^{-1}[u^\alpha S[L(\sum_{n=0}^{\infty} p^n U_n(x,t)) + (\sum_{n=0}^{\infty} p^n H_n(U(x,t)))]]) \ . \tag{8}$$

This is coupling of Sumudu transform and homotopy perturbation method using He's polynomials. By equating the coefficients of corresponding power of p on both sides, the following approximations are obtained as:

$$p^0 : U_0(x,t) = Q(x,t)\ , \tag{9}$$

$$p^1 : U_1(x,t) = -(S^{-1}[u^\alpha S[L(U_0(x,t)) + (H_0(U(x,t)))]])\ , \tag{10}$$

$$p^2 : U_2(x,t) = -(S^{-1}[u^\alpha S[L(U_1(x,t)) + (H_1(U(x,t)))]])\ , \tag{11}$$

$$p^3 : U_3(x,t) = -(S^{-1}[u^\alpha S[L(U_2(x,t)) + (H_2(U(x,t)))]]). \tag{12}$$

Proceeding in the same manner, the rest of the components $U_n(x,t)$ can be completely obtained, and the series solution is thus entirely determined. Finally we approximate the solution $U(x,t)$ by truncated series [15]

$$U(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N} U_n(x,t). \tag{13}$$

These series solutions generally converge very rapidly.

III. Applications:

In this section we apply this method for solving fractional initial-boundary value problem (FIBVP) and fractional linear inhomogeneous KDV *equation*.

Application 1: **The linear inhomogeneous fractional kdv equation.**

The discovery of solitary waves inspired scientists to conduct a huge size of research work to study this concept. Two Dutchmen Korteweg and Devries derived a nonlinear partial differential equation, well known by the KdV equation, to model the height of the surface of shallow water in the presence of solitary waves [18]. The KdV equation also describes the propagation of plasma waves in a dispersive medium.

We next consider the linear inhomogeneous fractional KDV equation:

$$D_t^\alpha U(x,t) + U_x(x,t) + U_{xxx}(x,t) = 2t\cos(x), t \succ 0, 0 \prec \alpha \le 1, x \in R \tag{14}$$

Subject to the initial condition $U(x,0) = 0$

We can solve Eq (14) by HPSTM by applying the Sumudu transform on both sides of (14), we obtain:

$$S[D_t^\alpha U(x,t)] + S[U_x(x,t) + U_{xxx}(x,t)] = S[2t\cos(x)]\ . \tag{15}$$

Using the property of the Sumudu transform, we have

$$S[U(x,t)] = U(x,0) - u^\alpha S[U_x(x,t) + U_{xxx}(x,t) - 2t\cos(x)]\,. \tag{16}$$

Now applying the Sumudu inverse on both sides of (16) we obtain:

$$U(x,t) = 2\cos(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - S^{-1}[u^\alpha S[U_x(x,t) + U_{xxx}(x,t)]] \tag{17}$$

Now, applying the classical homotopy perturbation technique, the solution can be expressed as a power series in p as given below:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t)\,. \tag{18}$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$).

By substituting from (18) into (17) and using HPM we get:

$$\begin{aligned}\sum_{n=0}^{\infty} p^n U_n(x,t) = 2\cos(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \\ pS^{-1}[u^\alpha S[(\sum_{n=0}^{\infty} p^n U_n(x,t))_x + (\sum_{n=0}^{\infty} p^n (U_n(x,t))_{xxx})]\end{aligned} \tag{19}$$

By equating the coefficient of corresponding power of p on both sides, the following approximations are obtained as:

$$p^0 : U_0(x,t) = 2\cos(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \tag{20}$$

$$p^1 : U_1(x,t) = 0, \tag{21}$$

$$p^2 : U_2(x,t) = 0 \ , \tag{22}$$

$$U_3(x,t) = 0, U_4(x,t) = 0,... \tag{23}$$

The HPSTM series solution is

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + ... \tag{24}$$

$$\therefore U(x,t) = 2\cos(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \tag{25}$$

For the special case $\alpha = 1$, we obtain

$$U(x,t) = t^2 \cos(x). \tag{26}$$

Which is the exact solution received by HAM [19] and VIM [20].

Application 2: Fractional Initial–Boundary value problem.

It was indicated that many phenomena of physics and engineering are expressed by partial differential equations PDEs. The PDE is termed a Boundary Value Problem (BVP) if the boundary conditions of the dependent variable u and some of its partial derivatives are often prescribed. However, the PDE is called an Initial Value Problem (IVP) if the initial conditions of the dependent variable u are prescribed at the starting time $t = 0$. Moreover, the PDE is termed Initial-Boundary Value Problem (IBVP) if both initial conditions and boundary conditions are prescribed. [21].

Let us consider the following time-fractional initial boundary value problem:

$$D_t^\alpha U(x,t) = U_{xx}(x,t) + xU_x(x,t) + U(x,t),\ 0 < \alpha < 1, t < 0\ , \tag{27}$$

$$U(x,0) = x\ , U_x(x,0) = 1\ , U(0,t) = 0 \tag{28}$$

By using HPSTM, we can solve Eq (27) by applying the Sumudu transform on both sides of (27), we obtain:

$$S[D_t^\alpha U(x,t)] = S[U_{xx}(x,t) + xU_x(x,t) + U(x,t)]. \tag{29}$$

Using the property of the Sumudu transform, we have

$$S[U(x,t)] = U(x,0) + u^\alpha S[U_{xx}(x,t) + xU_x(x,t) + U(x,t)]. \tag{30}$$

Now applying the Sumudu inverse on both sides of (30) we obtain:

$$U(x,t) = U(x,0) + S^{-1}[u^\alpha S[U_{xx}(x,t) + xU_x(x,t) + U(x,t)]\ . \tag{31}$$

Now, applying the classical homotopy perturbation technique, the solution can be expressed as a power series in p as given below:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t)\cdot \tag{32}$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$).

By substituting from (30) into (29) and using HPM we get:

$$\left.\begin{aligned}\sum_{n=0}^{\infty} p^n U_n(x,t) &= U(x,0) + S^{-1}[u^\alpha S[(\sum_{n=0}^{\infty} p^n U_n(x,t))_{xx} \\ &+ x(\sum_{n=0}^{\infty} p^n U_n(x,t))_x + (\sum_{n=0}^{\infty} p^n U_n(x,t))]]\end{aligned}\right\}. \tag{33}$$

By equating the coefficient of corresponding power of p on both sides, the following approximations are obtained as:

$$p^0 : U_0(x,t) = U(x,0) = x\ , \tag{34}$$

$$p^1 : U_1(x,t) = 2x\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{35}$$

$$p^2 : \quad U_2(x,t) = 4x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\ , \tag{36}$$

Similarity, we can get:

$$U_3(x,t) = 8x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},... \tag{37}$$

The HPSTM series solution is

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + ... \tag{38}$$

$$\therefore U(x,t) = x\sum_{n=0}^{\infty} \frac{(2t^{\alpha})^k}{\Gamma(\alpha k+1)}, \tag{39}$$

where $\sum_{k=0}^{\infty}\frac{(2t^{\alpha})^k}{\Gamma(\alpha k+1)} = E_{\alpha}(2t^{\alpha})$ is the famous Mittag–Leffler function.

i.e. Eq. (39) takes the form

$$U(x,t) = xE_{\alpha}(2t^{\alpha}). \tag{40}$$

For the special case $\alpha = 1$, we obtain

$$U(x,t) = xe^{2t}. \tag{41}$$

which is exactly the same solution as in [19] and [22].

IV. Conclusions

In this paper, the homotopy perturbation Sumudu transform method (HPSTM) is successfully applied for getting the analytical exact solution of the fractional initial-boundary value problem (IBVP) and the linear inhomogeneous fractional KDV equation. The results of the illustrated examples are in agreement with the results of the method presented in [19] and [22]. In conclusion, HPSTM is a very powerful and efficient method to find exact and approximate solutions as well as numerical solutions.

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